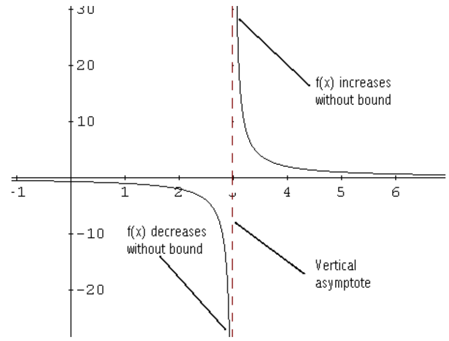
Unit 7

7.1 Operations with Rational Expressions

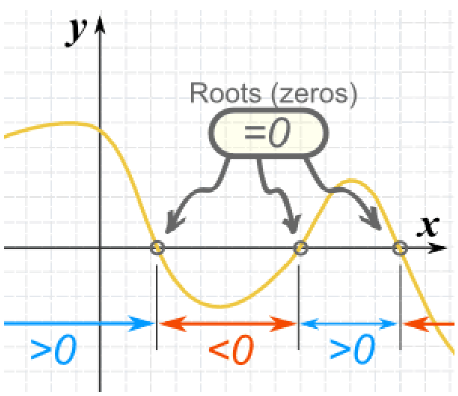
7.2 Rational Equations

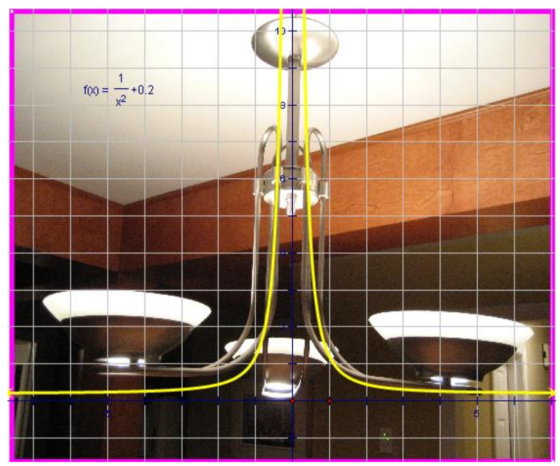
7.3 Rational Graphs and Asymptotes

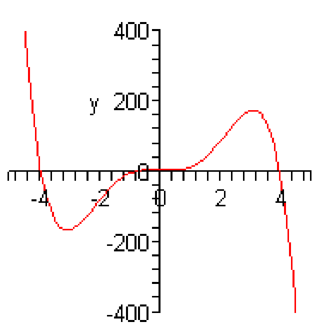
7.4 Higher Order Polynomials

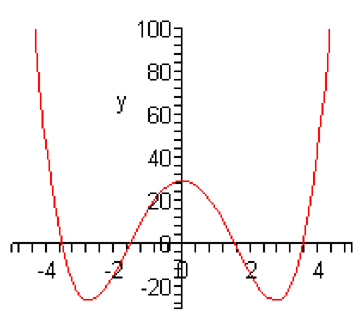


7.5 Polynomial, Rational, and Absolute Value Inequalities









7.1 Operations with Rational Expressions

A rational expression is a fraction or ratio of polynomials. Some examples are $\frac{4}{x+3}$ and $\frac{x^2+5}{2x-9}$. Operations on rational expressions follow the same rules as operations on common fractions. To reduce a rational expression, we use the fundamental principle of fractions.

Fundamental Principle of Fractions:

If we multiply or divide the numerator and denominator of a fraction by the same non-zero number, then we create a fraction which is equivalent to the original.

$$\frac{ac}{bc} = \frac{a}{b}, \qquad b, c \neq 0$$

To apply this to reducing rational expressions, the numerator and denominator must be written as a product of factors and then common factors from the numerator and denominator can be eliminated (cancelled out).

Examples:

1. Simplify: $\frac{4x^3y^2}{x^2y^2}$

$$\frac{4x^3y^2}{x^2y^2} = \frac{4x \cdot \cancel{x^2y^2}}{\cancel{x^2y^2}} = 4x$$

2. Simplify: $\frac{8x-4y}{4x}$

$$\frac{8x-4y}{4x} = \frac{\cancel{4}(2x-y)}{\cancel{4}x} = \frac{2x-y}{x}$$

3. Simplify: $\frac{x^2-16}{2x+8}$

$$\frac{x^2-16}{2x+8} = \frac{\cancel{(x+4)}(x-4)}{2\cancel{(x+4)}} = \frac{x-4}{2}$$

It is important to factor the numerator and denominator first, then divide out the common factors. Remember that we can not cancel terms!

Products of Rational Expressions:

To multiply two or more fractions, we multiply the numerators together and then the denominators together. When multiplying rational expressions, the numerators and denominators of all fractions should be factored first, all common factors should be divided out, then the fractions should be multiplied. The denominator is usually written as a product and not actually multiplied together.

Examples:

1. Multiply and simplify if possible: $\frac{10}{x+1} \cdot \frac{x-2}{5x-5}$

   First factor the numerators and denominators where possible. Then divide out common factors. Multiply the fractions.

$$\frac{10}{x+1} \cdot \frac{x-2}{5x-5} = \frac{2\cdot\cancel{5}}{x+1} \cdot \frac{x-2}{\cancel{5}(x-1)} = \frac{2}{x+1} \cdot \frac{x-2}{x-1} = \frac{2x-4}{(x+1)(x-1)}$$

2. Multiply and simplify if possible: $\frac{x+5}{x^2-1} \cdot \frac{4x-4}{x^2+x-20}$

   First factor the numerators and denominators where possible. Then divide out common factors. Multiply the fractions.

$$\frac{x+5}{x^2-1} \cdot \frac{4x-4}{x^2+x-20} = \frac{\cancel{x+5}}{(x+1)\cancel{(x-1)}} \cdot \frac{4\cancel{(x-1)}}{\cancel{(x+5)}(x-4)} = \frac{4}{(x+1)(x-4)}$$

Quotients of Rational Expressions:

To divide two rational expressions, we rewrite the problem as a multiplication where we multiply the first fraction by the reciprocal of the second fraction. Then factor all numerators and denominators where possible, divide out common factors, and then multiply the fractions.

Example:

Divide and simplify if possible: $\frac{3x+12}{x^2+3x+2} \div \frac{x^2-16}{6x+12}$

Rewrite the problem as a multiplication problem and then follow the steps we used to multiply fractions.

$$\frac{3x+12}{x^2+3x+2} \div \frac{x^2-16}{6x+12} = \frac{3x+12}{x^2+3x+2} \cdot \frac{6x+12}{x^2-16} = \frac{3\cancel{(x+4)}}{\cancel{(x+2)}(x+1)} \cdot \frac{6\cancel{(x+2)}}{\cancel{(x+4)}(x-4)} = \frac{18}{(x+1)(x-4)}$$

Adding and Subtracting Rational Expressions

To add fractions, the fractions must have a common denominator. Fractions with the same denominator are called like fractions and can be added or subtracted by adding or subtracting the numerators and keeping the same denominator.

Example:

Add and simplify if possible: $\frac{2x+5}{x-2}+\frac{x-4}{x-2}$

These fractions have the same denominators so the fractions can be added by adding the numerators and keeping the same denominator.

$$\frac{2x+5}{x-2}+\frac{x-4}{x-2}=\frac{3x+1}{x-2}$$

If the rational expressions do not have a common denominator, first factor the denominators, then find the lowest common denominator. The lowest common denominator (LCD) should be a polynomial of least degree that is divisible by each of the denominators. To find the LCD, factor each denominator and then include each different factor in the LCD as many times as it occurs in any one of the denominators. Then, each fraction (numerator and denominator) should be multiplied by the factors that are missing in the denominator of that fraction that are in the LCD. Once the fractions have a common denominator, then add or subtract the numerators and keep the same denominator. Simplify if possible.

Examples:

1. Add and simplify if possible: $\frac{2x+7}{x+3}+\frac{3x-4}{x-2}$

   The LCD will be the product of the 2 denominators (x+3)(x-2) since neither can be factored and they are not the same.

$$\frac{2x+7}{x+3}+\frac{3x-4}{x-2}=\frac{2x+7}{x+3}\cdot\frac{x-2}{x-2}+\frac{3x-4}{x-2}\cdot\frac{x+3}{x+3}=\frac{2x^2+3x-14}{(x+3)(x-2)}+\frac{3x^2+5x-12}{(x-2)(x+3)}=\frac{5x^2+8x-26}{(x+3)(x-2)}$$

2. Subtract and simplify if possible: $\frac{x+2}{3x+3}-\frac{2x-1}{x^2-1}$

   Factoring the denominators:

$$\frac{x+2}{3x+3}-\frac{2x-1}{x^2-1}=\frac{x+2}{3(x+1)}-\frac{2x-1}{(x+1)(x-1)}$$

   The LCD is 3(x+1)(x-1) since it must have every factor that appears in each denominator but we do not need to duplicate the (x+1) since it appears at most once in each denominator.

$$\frac{x+2}{3(x+1)}-\frac{2x-1}{(x+1)(x-1)}=\frac{x+2}{3(x+1)}\cdot\frac{(x-1)}{(x-1)}-\frac{2x-1}{(x+1)(x-1)}\cdot\frac{3}{3}=$$

$$\frac{x^2+x-2}{3(x+1)(x-1)}-\frac{6x-3}{3(x+1)(x-1)}=\frac{x^2-5x+1}{3(x+1)(x-1)}$$

Homework:

Simplify the following if possible:

1. $\frac{4x-12}{x^2+4x-21}$

2. $\frac{x^2-5x-14}{x^2-6x-16}$

3. $\frac{8x^2+20x}{2x+5}$

4. $\frac{15x+60}{9x^2-144}$

5. $\frac{2x^2-14x}{6x^2+24x}$

6. $\frac{12x^4y^2}{24xy}$

7. $\frac{50d^2}{60d^4}$

8. $\frac{6g(g-6)}{12g^2(g-6)}$

9. $\frac{(h+1)^2(h+2)}{3h(h-1)(h+1)}$

10. $\frac{10(j-1)^n}{j(j-1)^2}$

11. $\frac{t^2-9}{t^2-9t}$

12. $\frac{v^2-2v+1}{4v^2-4}$

Multiply or divide, simplify if possible.

13. $\frac{5y-10}{2x+2} \cdot \frac{x+1}{3y-6}$

14. $\frac{(x+1)(x-3)}{2x+5} \cdot \frac{x(2x+5)}{4(x-3)}$

15. $\frac{y^2+6y+5}{7y^2-63} \cdot \frac{7y+21}{(y+5)^2}$

16. $\frac{2x^2-2x}{4x^2-1} \cdot \frac{10x-5}{4x}$

17. $\frac{6a+24}{5} \div \frac{4a+16}{35}$

18. $\frac{x^2-16x+64}{3x^2-192} \div \frac{3x^2-30x+48}{x^2+6x-16}$

19. $\frac{a^2+2a-15}{a^2+3a-10} \div \frac{a^2-9}{a^2-9a+14}$

20. $\frac{9x^2-25}{2x-2} \div \frac{6x-10}{x^2-1}$

Add or subtract, simplify if possible.

21. $\frac{m+3}{m-4} - \frac{m+17}{m-4}$

22. $\frac{11p+3}{p-1} + \frac{4p+1}{p-1}$

23. $\frac{4x+7}{x+2} + \frac{x+3}{x+2}$

24. $\frac{t-9}{3t} - \frac{4t-3}{3t}$

25. $\frac{1}{y}-\frac{3}{x}$

26. $\frac{11}{p}+\frac{4}{p-1}$

27. $\frac{3}{2a}+\frac{a-2}{3}$

28. $\frac{3}{m-4}-\frac{7}{m+1}$

29. $\frac{t}{x+y}-\frac{2t}{x-y}$

30. $\frac{2}{cd}+\frac{4}{c^2}$

31. $\frac{m}{2m-4}-\frac{m+1}{2m-6}$

32. $\frac{10}{3g-2}+\frac{g+1}{6g^2-4g}$

33. $\frac{2x}{x^2-y^2}-\frac{y}{(x+y)^2}$

34. $\frac{5}{x^2-9}+\frac{2x}{x^2+4x+3}$

35. $\frac{2}{y+3}-\frac{5}{y+4}$

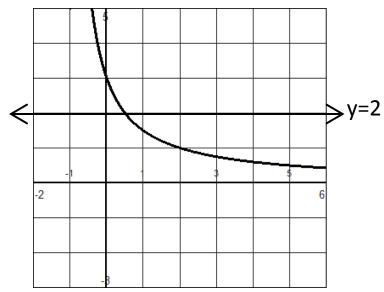
36. $\frac{-11}{5t-25}+\frac{4}{3t-15}$

7.2 Rational Equations

To solve a rational equation, we first clear the fractions from the equation by multiplying both sides of the equation by the common denominator, then solving the resulting equation. When solving rational equations, we must check for extraneous solutions. Rational equations can be solved numerically, graphically, and algebraically.

Examples:

1. Solve $\frac{3}{x+1} = 2$ graphically given the graph of y = $\frac{3}{x+1}$ as shown.



   To solve graphically, we need to draw the graph of the line y = 2 on the graph and find the x-value of the intersection point. The x-value of the intersection point is x = ½.

2. Solve $\frac{3}{x+1} = 2$ algebraically.

   To solve we multiply both sides of the equation by the denominator.

$(x+1)\cdot\frac{3}{x+1} = 2(x+1)$	On the left side, the (x+1) terms divide out.
3 = 2x+2	Distributing the right side of the equation
1 = 2x	Subtracting 2 from both sides
½ = x	Dividing both sides of the equation by 2

3. Solve $\frac{4}{2x-1} = \frac{5}{x+3}$ algebraically.

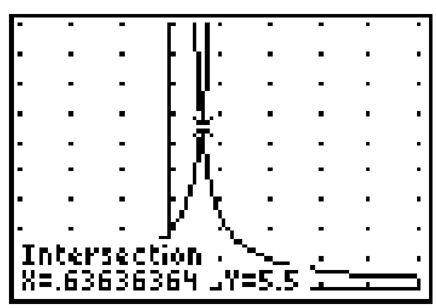
   The LCD is (2x-1)(x+3). To solve we clear the fractions by multiplying by the LCD.

   $(2x-1)(x+3)\frac{4}{2x-1} = \frac{5}{x+3}(2x-1)(x+3)$ Dividing out like terms

4x + 12 = 10x – 5	Multiplying out both sides of the equation
-6x = -17	Subtracting 10x and 12 from both sides of the equation
x = 17/6	Dividing both sides by -6

4. Solve $\frac{5}{3x-1} = \frac{2}{1-x}$ graphically.

   On the graphing calculator, graph $y_1 = \frac{5}{3x-1}$ and $y_2 = \frac{2}{1-x}$. Choose a window where you can see the intersection point of the two graphs. Use the intersection command to find the intersection point. The solution is the x-value of the intersection point which is x = 0.6364.



A rational function is undefined for any value which makes its denominator equal to zero. When solving rational equations, we must check to be sure the solution that is found does not make the rational function undefined.

Example:

Solve $\frac{2}{x-2} = \frac{x}{x-2} + 2$ algebraically.

The LCD is x-2. Multiply both sides of the equation by the LCD.

$(x-2)\frac{2}{x-2} = (x-2)\left[\frac{x}{x-2} + 2\right]$

2 = x + 2(x – 2)	Simplifying both sides
2 = x + 2x – 4	Distribute
6 = 3x	Add 4 to both sides
x = 2	Divide both sides of the equation by 3

But x = 2 makes the denominator of the rational expression in the equation zero so it is an extraneous solution. Therefore, there is no solution to this equation.

Now let's look at some applications involving rational equations. The most common applications are proportions, motion problems and work problems.

Proportion Problems: Proportion problems involve two ratios which are set equal to each other. When setting up the ratios, the numerators of each ratio should represent the same object under discussion and the denominators must represent the same item in each fraction. For example, if the ratio of green to red apples is 2 to 1, you can write G/R = 2/1. Both numerators must represent the same type of object, and both denominators must represent the same type. Here, G and 2 stand for green apples, and R and 1 stand for red apples.

Example:

Debra has a candy apple red, convertible Mustang GT. In the last year she has driven 11,600 miles in it and she has changed the oil four times. If she continues maintaining her car at this rate, and she only drives 5,700 miles next year, about how many times will she change the oil?

The two quantities in this problem are the miles driven and the number of oil changes. It does not matter which you choose to be in the numerator or denominator but you must remain consistent for both ratios. Choosing the number of oil changes for the numerator gives:

$$\frac{4}{11600} = \frac{c}{5700}$$

Cross-multiplying yields:
22800 = 11600c
c = 1.9655 or approximately 2 oil changes.

Work problems usually involve two people or objects completing a task individually and then the question is how long it will take them to complete the task if they work together.

Work Problems: If people *M* and *N* work together on a project, then $\frac{1}{m}+\frac{1}{n}=\frac{1}{t}$, where

m is the time needed for person *M* to complete the project,
n is the time need for person *N* to complete the project, and
t is the time needed for *M* and *N* to complete the project together.

Example:

Steve and Janet are going to paint the picket fence that surrounds their house today. Steve can paint the fence alone in 12 hours. Janet can paint the fence alone in 9 hours. How long will it take them to paint the fence together?

To solve this problem, we should first **define a variable** that represents the quantity that we are looking for:
Let *t* represent the time, in hours, that it takes Steve and Janet to paint the fence together.
To find an equation that represents the given scenario, let's determine how much of the fence Steve and Janet paint, respectively.
Since it takes Steve 12 hours to paint the fence, each hour he paints 1/12 of the fence. Since it takes Janet 9 hours to paint the fence, each hour she paints 1/9 of the fence. Since it takes t hours for them to paint the fence together, each hour they paint 1/t of the fence.
Amount painted by Steve + Amount painted by Janet = Amount painted together
1/12 + 1/9 = 1/t
To solve we multiply both sides of the equation by the common denominator 36t.
(36t)(1/12 + 1/9) = (1/t)(36t)
3t + 4t = 36
7t = 36
t = 36/7 hours or t = 5.14 hours. Note that the solution should be less time than it takes the slower person to complete the task.

Motion Problems: Motion problems involve the formula Distance = Rate ·Time. When an object is moving on water or through the air the object's rate depends not only on its rate but also on the speed of the current or the speed of the wind. If an object is going downstream, then the current increases the object's overall speed. Upstream the current is pushing against the object and slowing it down.

Example:

I went canoeing on the Econ River last Saturday. I traveled 12 miles downstream and then turned around and traveled the same 12 miles upstream. The trip took me a total of 4 hours. If my canoe travels 7 mph (miles per hour) in still water, what is the speed of the current in the river?

Since the total time spent canoeing was 4 hours, we can obtain an equation by adding the time spent canoeing upstream with the time spent canoeing downstream and setting this sum equal to 4. Since we are asked to find the speed of the current in the Econ River, let's let *s* represent the speed of the river (in mph).

Since we want to add the times spent canoeing up and downstream, we will want to utilize the formula $d=r\cdot t$ to obtain expressions for the upstream and downstream times.

When I canoed downstream, I traveled 12 miles (so d = 12) and the rate that I traveled was (7 + s) mph (the sum of my speed in still water and the rate of the current). So the time spent traveling downstream $t = \frac{12}{7+s}$. When I canoed upstream, I traveled 12 miles and the rate that I traveled was (7 – s) mph (the difference of my speed in still water and the rate of the current). So the time spent traveling upstream $t = \frac{12}{7-s}$.
Thus, if we add the time spent traveling downstream and the time spent traveling upstream, we should get 4 hours: $\frac{12}{7+s} + \frac{12}{7-s} = 4$
To solve this equation we need to multiply by the LCD (7+s)(7-s).

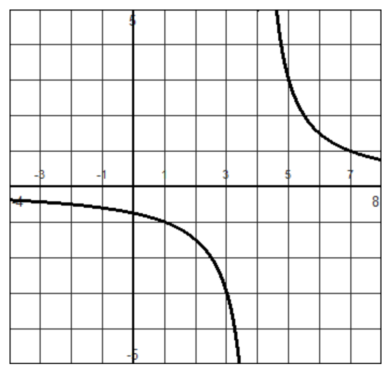
$$(7+s)(7-s)\frac{12}{7+s} + (7+s)(7-s)\frac{12}{7-s} = 4(7+s)(7-s)$$

12(7-s) + 12(7+s) = 4(7+s)(7-s)
84 – 12s + 84 + 12s = 196 – $4s^2$
168 = 196 – $4s^2$
-28 = -$4s^2$
7 = s^2
s = $\sqrt{7}$ which is approximately 2.65 mph (the speed of the current).

Homework:

1. Given the graph of y = $\frac{3}{x-4}$ as shown, solve the equations:
   A) $\frac{3}{x-4} = 3$

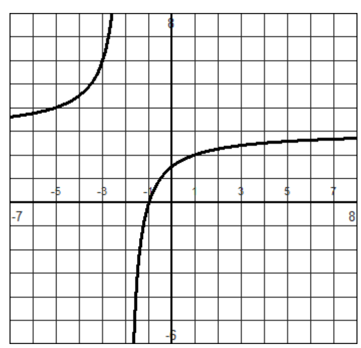
   B) $\frac{3}{x-4} = -1$



2. Given the graph of y = $\frac{3x+3}{x+2}$ as shown, solve the equations:
   A) $\frac{3x+3}{x+2} = 6$

   B) $\frac{3x+3}{x+2} = 2$

   C) $\frac{3x+3}{x+2} = -3$



3. Use your graphing calculator to solve the equation $\frac{7}{x+1} = \frac{-3}{x-2}$.

4. Use your graphing calculator to solve the equation $\frac{5}{2x+1} = \frac{-3}{x+3}$.

5. $\frac{4}{x-3}=\frac{3}{2x+1}$

6. $\frac{c-5}{c+1}=17$

7. $x-\frac{12}{x}=4$

8. $\frac{-5}{5x-2}=\frac{-3}{3x+1}$

9. $\frac{7}{4x+1}=\frac{3}{2x+5}$

10. $\frac{4}{5}=\frac{3x-7}{2x+1}$

11. $\frac{4}{3-t}+\frac{4}{3+t}=10$

12. $\frac{3.75}{7}=\frac{2}{x}$

For 13 – 25, set up a proportion and solve.

13. Hilary bought 12 souvenirs during 8 days of vacation. Assuming she continues to buy them at the same rate, how many days will Hilary have to spend on vacation before she will have bought a total of 20 souvenirs?

14. Roscoe jarred 6 liters of jam after 4 days. Assuming he continues working at the same speed, how many days did Roscoe spend making jam if he jarred 15 liters of jam?

15. Renee consumed a total of 15 liters of water over 3 days. How many liters of water has Renee consumed after 4 days?

16. Steve likes to push his endurance to the limit. He knows his body temperature will drop 2.5 degrees every 7 seconds that he is in the ice cold water. He also knows that hypothermia will set in if his body temperature drops **below** 65 degrees Fahrenheit, and he will die. How long can Steve stay in the water and not go UNDER 65 degrees if his body temperature is 98.6 degrees when he jumps into the lake?

17. Carol spends 17 hours in a 2-week period practicing her culinary skills. How many hours does she practice in 5 weeks?

18. In the typing world, 80 words per minute is considered acceptable. How many words per 30 minutes is this?

19. In the year 2000, there were 8.7 deaths per 1000 residents in the United States. If there were 281,421,906 residents in the U.S. during 2000, how many people died that year?

20. In a shipment of 400 parts, 15 are found to be defective. How many defective parts should be expected in a shipment of 1000?

21. A snowstorm dumped 18 inches of snow in a 12-hour period. How many inches were falling per hour?

22. Mary can read 22 pages in 30 minutes. How long would it take her to read a 100 page book? Write your answer in hours and minutes and round to the nearest minute, if needed.

23. The chance of a woman getting breast cancer in her lifetime is 1 out of 8. At this rate, how many women in a classroom of 32 women would be expected to come down with breast cancer in her lifetime?

24. To determine the number of deer in a forest, a forest ranger tags 280 and releases them back into the forest. Later, 405 deer are caught, out of which 45 of them are tagged. Estimate how many deer are in the forest.

25. If five pounds of potatoes cost $2.99, how many pounds of potatoes could you buy for $8.50?

26. When the same number is added to the numerator and denominator of the fraction $\frac{3}{5}$, the result is $\frac{7}{9}$. What is the number that is added?

27. Bill, working alone, can pour a concrete walkway in 6 hours. Greg, working alone, can pour the same walkway in 4 hours. How long will it take both of them to pour the concrete walkway working together?

28. An inlet pipe can fill a water tank in 8 hours and an outlet pipe can drain the tank in 10 hours. If both pipes are open, how long will it take to fill the tank?

29. After painting the fence Steve and Janet realize that their house also needs painting. But Steve will have to do the painting himself since Janet has injured her shoulder. Before getting started, Steve wants to find out how long the project will take. He determines that it would take them 9 days to paint the house together (if they were both healthy) and that it would take Janet (when healthy) 16 days to paint the house alone. How long will it take Steve to paint the house alone?

30. It takes John 3 hours longer than Pete to complete a certain job. Working together, both can complete the job in 2 hours. How long does it take each person to complete the job working alone?

31. One pipe can drain a pool in 12 hours. Another pipe can drain the pool in 15 hours. How long does it take both pipes working together to drain the pool?

32. A faucet can fill a bathroom sink in 1 minute. The drain can empty the sink in 2 minutes. If both the faucet and drain are open, how long will it take to fill the sink?

33. A faucet can fill a bathtub in 6 ½ minutes. The drain can empty the tub in 8 ¼ minutes. If both the faucet and drain are open, how long will it take to fill the bathtub?

34. An inlet pipe can fill a tank in 5 hours. An outlet pipe can empty the tank in 4 hours. If both pipes are open, can the tank be filled? Explain.

35. A delivery boy, working alone, can deliver all his goods in 6 hours. Another delivery boy, working alone, can deliver the same goods in 5 hours. How long will it take the boys to deliver all the goods working together?

36. A Space Shuttle astronaut can perform a certain experiment in 2 hours. Another Space Shuttle astronaut who is not as familiar with the experiment can perform it in 2 ½ hours. Working together, how long will it take both astronauts to perform the experiment?

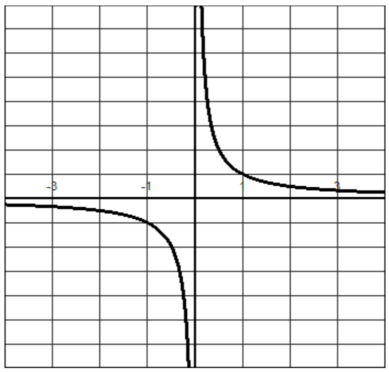
37. On Tuesday and Wednesday of last week I left home for school at the same time. On Tuesday I drove 40 mph (miles per hour) on my way to school and arrived 3 minutes late. On Wednesday I drove 50 mph on my way to school and arrived 2 minutes early. How far do I live from work?

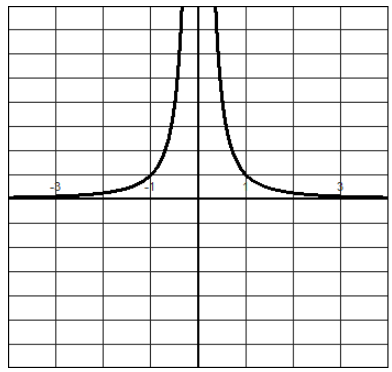
38. James' car is traveling 30 km/hr faster than Jody's car. During the time it takes Jody to go 250 km, James goes 420 km. Find their speeds.

39. A fully loaded train travels 16 km/hr slower than an empty train. The loaded train travels 440 km in the same time it takes the empty train to travel 600 km. Find the two speeds.

7.3 Rational Graphs and Asymptotes

Recall the basic shapes of rational graphs. They were the graphs of y = $\frac{1}{x}$ and y = $\frac{1}{x^2}$ which are shown below.





In this section, we will investigate other rational graphs and their applications.

A **rational function** is one of the form f(x) = $\frac{P(x)}{Q(x)}$ where P(x) and Q(x) are polynomials and Q(x) is not zero.

A **vertical asymptote** occurs at x = a if Q(a) = 0 but P(a) ≠ 0. A vertical asymptote is an invisible line that the graph approaches but does not touch.

If at x = a, both Q(a) = 0 and P(a) = 0, then the graph has a **hole (a missing point)** in the graph.

Therefore, all values where the rational function is undefined will either be vertical asymptotes or holes in the graph.

Examples:

Find all vertical asymptotes or holes in the graphs of the following functions.

1. f(x) = $\frac{2x+1}{x-9}$
2. h(x) = $\frac{x+1}{x^2-3x-4}$
3. g(x) = $\frac{3x^2+1}{x^2-4}$

Solutions:

1. We need to find the values of x where the function is undefined. These occur where the denominator is zero.

   x – 9 = 0 at x = 9 and since x = 9 does not make the numerator zero, x = 9 is a vertical asymptote.

2. We need to find the values of x where the function is undefined. These occur where the denominator is zero.

   $x^2 - 3x - 4 = 0$ when $(x - 4)(x + 1) = 0$

   This occurs at x = 4 and x = -1. Checking to see if the numerator is zero at these values we find that x = -1 makes the numerator zero and x = 4 does not. Therefore, x = -1 is a hole in the graph and x = 4 is a vertical asymptote.

3. We need to find the values of x where the function is undefined. These occur where the denominator is zero.

   $x^2 - 4 = 0$ at $x = \pm 2$ and since neither of these makes the numerator zero, x = 2 and x = -2 are both vertical asymptotes.

If we want to find out the vertical position of the hole in the graph so that we can plot the open circle to show the hole, then this is sometimes possible by simplifying the function and then evaluating the new function at the x-coordinate of the hole. If the function can not be simplified

Example:

Find the coordinates of the hole in the graph of h(x) = $\frac{x+1}{x^2-3x-4}$.

Solution: From Example 2 above, we found that the h(x) = $\frac{x+1}{x^2-3x-4}$ had a hole at x = -1. To plot the open circle for the hole in the graph, factor the function and simplify.

$$\frac{x+1}{x^2-3x-4} = \frac{x+1}{(x-4)(x+1)} = \frac{1}{x-4}$$

Defining g(x) = $\frac{1}{x-4}$ will give a new function which has the same graph as the original function but without the hole at x = -1. If we plug in x = -1 to g(x), then we find g(-1) = $-\frac{1}{5}$. Therefore, the hole in the graph of h(x) will have a vertical position of $-\frac{1}{5}$. It is important to remember that a hole is not a point on the graph because it occurs at a value which is not in the domain of the function. This technique just allows you to find where to draw the open circle to represent the hole.

Some rational functions have horizontal asymptotes. A **horizontal asymptote** is an invisible horizontal line that the ends of the graph approach. This means that the ends of the graph will flatten out near the horizontal asymptote. To determine whether the graph will have a horizontal asymptote, we compare the degree of the numerator and the degree of the denominator. There are three possibilities:

1. If the degree of the numerator is larger than the degree of the denominator, then there will not be a horizontal asymptote.

2. If the degree of the numerator is smaller than the degree of the denominator, then there is a horizontal asymptote at y = 0.

3. If the degree of the numerator is equal to the degree of the denominator, then there is a horizontal asymptote at the ratio of the leading coefficients of the numerator and denominator.

Examples:

Determine whether or not the graphs have a horizontal asymptote and if so, where the asymptote is located.

1. f(x) = $\frac{2x+1}{x-9}$

2. h(x) = $\frac{x+1}{x^2-3x-4}$

3. g(x) = $\frac{3x^2+1}{5x^2-4}$

4. F(x) = $\frac{2x^2+1}{x+2}$

Solutions:

1. For f(x) = $\frac{2x+1}{x-9}$, the degree of the numerator is 1 and the degree of the denominator is 1, so the graph will have a horizontal asymptote (the degrees are equal). The horizontal asymptote will occur at the ratio of the leading coefficients so the horizontal asymptote is y = 2.

2. For h(x) = $\frac{x+1}{x^2-3x-4}$, the degree of the numerator is 1 and the degree of the denominator is 2, so the graph will have a horizontal asymptote (the degree of the numerator is smaller than the degree of the denominator). The horizontal asymptote will occur at y = 0.

3. For g(x) = $\frac{3x^2+1}{5x^2-4}$, the degree of the numerator is 2 and the degree of the denominator is 2, so the graph will have a horizontal asymptote (the degrees are equal). The horizontal asymptote will occur at the ratio of the leading coefficients so the horizontal asymptote is y = 3/5.

4. For F(x) = $\frac{2x^2+1}{x+2}$, the degree of the numerator is 2 and the degree of the denominator is 1, so the graph will not have a horizontal asymptote (the degree of the numerator is larger than the degree of the denominator).

As mentioned above, if the degree of the numerator is larger than the degree of the denominator, then there will not be a horizontal asymptote. But, if the degree of the numerator is only one degree larger than the degree of the denominator, there will be a **slant (oblique) asymptote**.

Just as a horizontal asymptote is a horizontal line, and a vertical asymptote is a vertical line, a slant asymptote is a slanted line typically written in the format: $y = mx + b$. To find the slant asymptote of your graph, you will divide the numerator by the denominator. This typically means you must perform polynomial long division or synthetic division. The slant asymptote is the quotient you obtain from the polynomial division without the remainder.

Examples:

1. Find the slant asymptote: $y = \frac{x^2-9x-10}{x+1}$.
   Using synthetic division:

$-1\rfloor$	1	-9	-10
		-1	10
	1	-10	0

   Therefore, the slant asymptote is y = x – 10.

2. Find the slant asymptote: $y = \frac{2x^2+5x-7}{x+3}$
   Using synthetic division:

$-3\rfloor$	2	5	-7
		-6	3
	2	-1	-4

   Therefore, the slant asymptote is y = 2x – 1 (ignoring the remainder of -4).

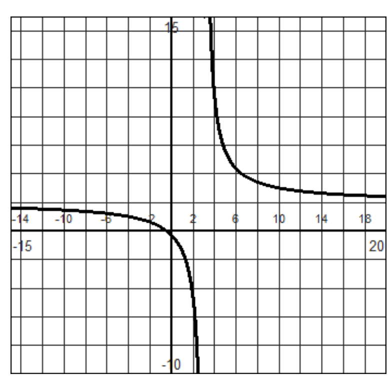
When sketching a rational function, we can use the asymptotes and the intercepts to help sketch the graph.

Examples:

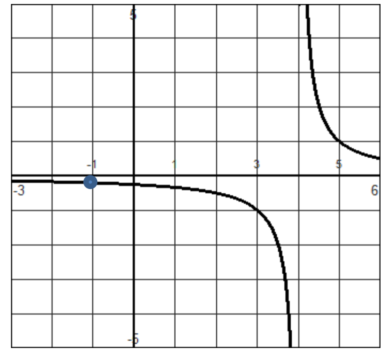
1. Sketch f(x) = $\frac{2x+1}{x-3}$.
2. Sketch h(x) = $\frac{x+1}{x^2-3x-4}$.
3. Sketch F(x) = $\frac{2x^2+1}{x+2}$.

Solutions:

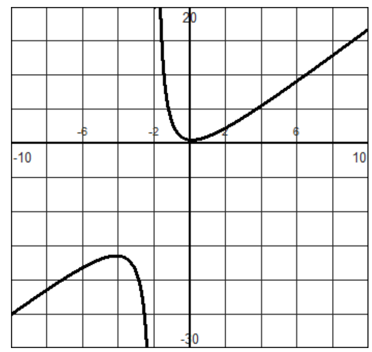
1. f(x) = $\frac{2x+1}{x-3}$ has a vertical asymptote at x = 3 and a horizontal asymptote at y = 2. It does not have any holes. It has an x-intercept at (-1/2, 0) and a y-intercept at (0, -1/3).



2. h(x) = $\frac{x+1}{x^2-3x-4}$ has a vertical asymptote at x = 4, a hole at x = -1 and a horizontal asymptote at y = 0 . The y-intercept is at (0, -1/4) and there is no x-intercept.



3. F(x) = $\frac{2x^2+1}{x+2}$ has a vertical asymptote at x = -2 and no horizontal asymptote. Instead, it has a slant asymptote at $y = 2x - 4$. It also has no x-intercepts and an y-intercept at (0, 1/2).



Rational functions can be used in applications involving rate/distance, population growth, average cost and others.

Examples:

1. A school of tuna is migrating a distance of 150 miles at an average speed of 34 miles per hour in still water. The fish are swimming against a current of c miles per hour.

   A. Complete the table which gives the travel time for the given current speeds.

c	0	4	8	20	24	34
t						

   B. Express the travel time, t, in as a function of the speed of the current, c.

   C. What happens to the travel time as the speed of the current increases?

   D. Sketch a graph in an appropriate window for this function.

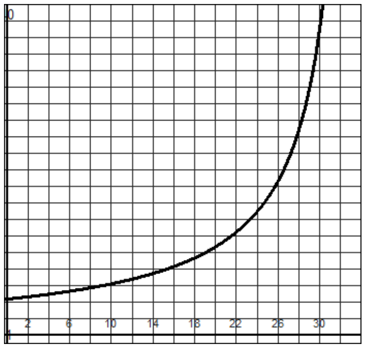
Solution:

A.

c	0	4	8	20	24	34
t	4.412	5	5.769	10.714	15	undefined

B. t = $\frac{150}{34-c}$

C. The travel time increases as the speed of the current increases because the speed the tuna are traveling is decreasing so it takes longer to reach their destination.

D. 

Notice that this graph has a vertical asymptote at c = 34 which means that when the current is 34 miles per hour, the tuna will never get to the destination.

2. A club decides to sell t-shirts as a fundraiser. The club spends an initial $50 on designing the shirts and setting up the printing process. In addition, each shirt costs $4 for the labor and materials.

A. Express the average cost per t-shirt as a function of the number of shirts that are made.

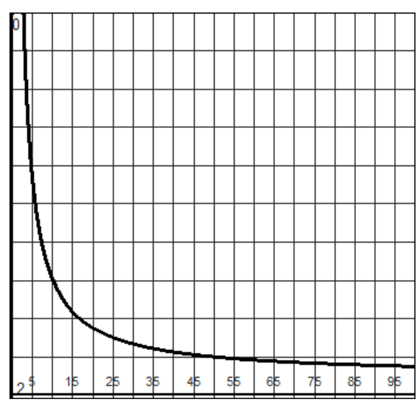
B. Sketch a graph and discuss what happens to the average cost as the number of shirts produced increases.

Solution:

A. The average cost is the total cost divided by the number of shirts. The total cost is 50 + 4x where x is the number of shirts.

$C(x) = \frac{50 + 4x}{x}$

B. Notice that as the number of t-shirts increases, the graph will approach the horizontal asymptote. The degree of the numerator and denominator are equal so the horizontal asymptote is C = 4. This means that the average cost per shirt is approaching $ 4 as they make more shirts.



Homework:

For the following equations:

A) Find any vertical asymptotes and holes of the function.

B) Find any horizontal or slant asymptotes if they exist.

C) Find the x-intercepts and y-intercepts.

D) Use the asymptotes and intercepts to help you sketch the graph.

1. $f(x) = \frac{x+4}{2x^2-x}$

2. $g(x) = \frac{4x^2+4}{2x^2-x-1}$

3. $f(x) = \frac{x+4}{x^2-16}$

4. $h(x) = \frac{x-5}{(x+1)(x-2)}$

5. $f(x) = \frac{x^2+4}{x-3}$

6. $g(x) = \frac{3x-6}{x^2-3x+2}$

7. $H(x) = \frac{x+4}{x+2}$

8. $f(x) = \frac{x^2-1}{3x-9}$

For 9 - 10: On the web site http://www.eren.doe.gov/consumerinfo/energy_savers/appliancesbody.html there is a list of yearly cost for electricity for common household appliances.

Appliance	Average Cost/year in electricity
Home Computer	\$9
Television	\$13
Microwave	\$13
Dishwasher	\$51
Clothes Dryer	\$75
Washing Machine	\$79
Refrigerator	\$92

9. Assume a new refrigerator costs \$ 750.
   a) Determine the total annual cost for a refrigerator that lasts for 15 years. Assume the only costs associated with the refrigerator are its purchase cost and electricity.
   b) Develop a function that gives the annual cost of a refrigerator as a function of the number of years you own the refrigerator.
   c) Sketch a graph of that function. What is an appropriate window?
   d) Since this is a rational function, determine the asymptotes of this function.
   e) Explain the meaning of the horizontal asymptote in terms of the refrigerator.
   f) If a company offers a refrigerator that costs \$1200, but says that it will last at least twenty years, is the refrigerator worth the difference in cost?

10. Assume a new washing machine costs \$ 560.
    a) Determine the total annual cost for a washing machine that lasts for 9 years. Assume the only costs associated with the washing machine are its purchase cost and electricity.
    b) Develop a function that gives the annual cost of a washing machine as a function of the number of years you own the washing machine.
    c) Sketch a graph of that function. What is an appropriate window?
    d) Since this is a rational function, determine the asymptotes of this function.
    e) Explain the meaning of the horizontal asymptote in terms of the washing machine.

11. The function $C(t) = \dfrac{5t}{0.01t^2 + 3.3}$ describes the concentration of a drug in the blood stream over time. In this case, the medication was taken orally. C is measured in micrograms per milliliter and t is measured in minutes.

    a) Sketch a graph of the function over the first two hours after the dose is given. Label axes.
    b) Determine when the maximum amount of the drug is in the body and the amount at that time.
    c) Explain within the context of the problem the shape of the graph between taking the medication orally ($t = 0$) and the maximum point. What does the shape of the graph communicate between the maximum point and two hours after taking the drug?
    d) What are the asymptotes of the rational function $C(t) = \dfrac{5t}{0.01t^2 + 3.3}$? What is the meaning of the asymptotes within the context of the problem?

12. Your family is traveling in your car for 100 miles.
    a) How long will the trip take if you average 30 miles per hour, 55 miles per hour, or 65 miles per hour?
    b) Write a function that describes the time it takes to make this trip as a function of your speed. Identify the meaning of the variables.
    c) Graph this function. Show asymptotes.
    d) What does this graph tell you about the time it will take you to travel depending on the speed of the car?

13. A club decides to sell t-shirts as a fundraiser. The club spends an initial \$80 on designing the shirts and setting up the printing process. In addition, each shirt costs \$ 6 for the labor and materials.

    a) Express the average cost per t-shirt as a function of the number of shirts that are made.

    b) Sketch a graph and discuss what happens to the average cost as the number of shirts produced increases.

14. The cost, in thousands of dollars, for extracting p percent of an ore from a mine is given by the equation C(p) = $\frac{275p}{100-p}$.

a) What is the domain of C(p)?

b) Find the cost of extracting 30% of the ore. What is the cost of extracting 50% of the ore?

c) Graph C(p) for $0 \leq p \leq 100$.

d) What percent of the ore can be extracted for $500,000?

e) The graph has a vertical asymptote. What is it? What does it represent in this problem?

15. The total cost of producing n calculators is 25000 + 15n.

a) Express the average cost per calculator as a function of the number of calculators that are produced.

b) Graph the function in an appropriate viewing window.

c) How many calculators should be produced so that the cost per calculator is $22 per calculator?

d) Find the horizontal asymptote. What does it represent in the context of this problem?

7.4 Higher Order Polynomials

We have investigated linear functions and quadratic functions in detail. In this section, we investigate the graphs of polynomial functions of degree 3 or more. To draw a complete graph of higher order polynomials requires calculus so this section focuses on being able to draw a rough sketch and understanding the basic shapes of the graphs.

The end behavior of polynomial functions is determined by the degree of the polynomial and the leading coefficient.

The **leading coefficient** of a polynomial function is the coefficient of the term of highest degree. If the polynomial is in descending order, then the leading coefficient is the coefficient of the first term. For example, the leading coefficient of $7x^4 - 5x^3 + x - 11$ is 7 and the leading coefficient of $4x - 5x^4 + x^8$ is 1 (the coefficient of x^8).

The **degree of a polynomial** is the highest degree of any one of its terms. For instance, the degree of $7x^4 - 5x^3 + x - 11$ is 4 and degree of $4x - 5x^4 + x^8$ is 8.

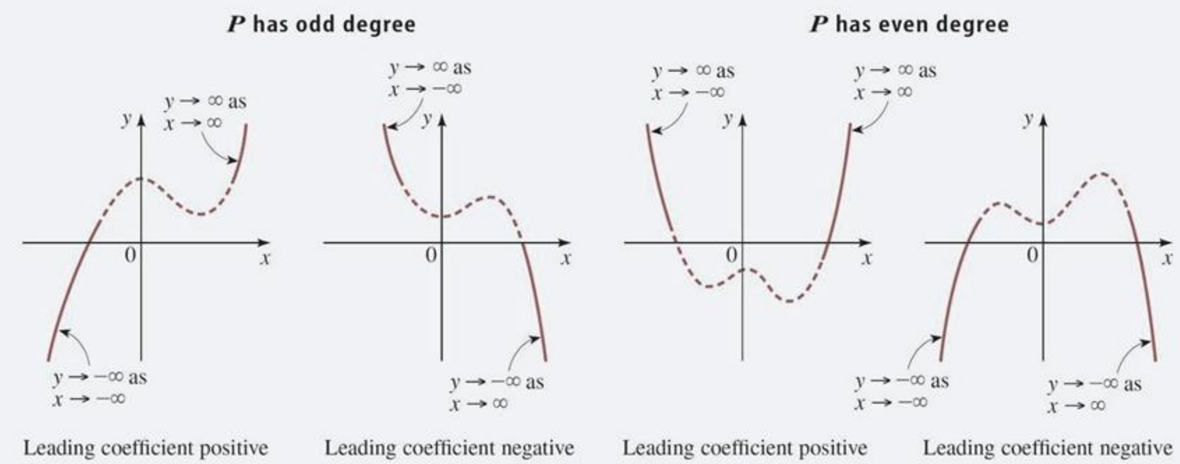
The **end behavior of a polynomial** is the behavior of the ends of the graphs over the long term. In other words, it describes the direction that the ends of the graph point. For example, the end behavior of a parabola $f(x) = x^2$ is that both ends of the graph point upward. The end behavior of all polynomial functions is summarized in the chart below.

End Behavior:

	Even degree	Odd Degree
Positive leading coefficient	Both ends up	Left down, Right up
Negative leading coefficient	Both ends down	Left up, Right down

END BEHAVIOR OF POLYNOMIALS

The end behavior of the polynomial $P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ is determined by the degree n and the sign of the leading coefficient a_n, as indicated in the following graphs.



Examples:

For the following polynomial functions, identify the degree, leading coefficient and the end behavior of the graph.

1. $P(x) = 6x^7 + 5x^2 – 9x + 1$
2. $P(x) = -11x^3 – 2x^2 – x$
3. $P(x) = ½\, x^6 – 3x^5 + 4x^2 – 9$

Solutions:

1. The degree is 7 (odd) and the leading coefficient is 6 (positive). The end behavior is left side down and right side up.
2. The degree is 3 (odd) and the leading coefficient is -11 (negative). The end behavior is left side up and right side down.
3. The degree is 6 (even) and the leading coefficient is ½ (positive). The end behavior is both sides up.

We usually look at polynomial functions in factored form in order to sketch the graphs. This allows us to identify the x-intercepts by using the zero-factor principle. A polynomial of degree n can have at most n x-intercepts.

Examples:

Find the x-intercepts and the degree of the following polynomials.

1. $P(x) = (x + 2)(x – 3)(x + 4)$
2. $P(x) = x(x – 5)(x – 7)^2$
3. $P(x) = -x^3(x – 1)^2$

Solutions:

1. Setting each factor equal to zero, we find the x-intercepts are (-2, 0), (3, 0), and (-4, 0). The degree is 3.
2. Setting each factor equal to zero, we find the x-intercepts are (0, 0), (5, 0) and (7, 0). The degree is 4.
3. Setting each factor equal to zero, we find the x-intercepts are (0, 0) and (1, 0). The degree is 5.

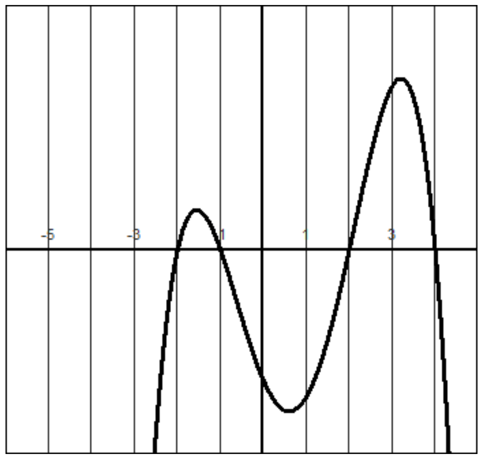
Combining the two ideas of end behavior with the x-intercepts allows us to draw a rough sketch of the graph. We will not be able to tell how high or low the maximum and minimum points are so we do not label the vertical axis with a scale. The graphs should be smooth curves.

Example:

Given P(x) = -(x + 2)(x+1)(x – 2)(x – 4), find the end behavior, the x-intercepts, and sketch the graph.

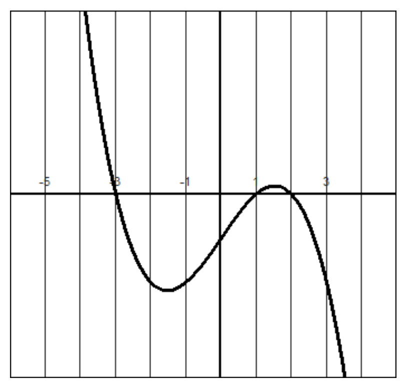
Solution:

The leading coefficient is negative and the degree is even so the end behavior is both ends down. The x-intercepts are (-2, 0)(-1, 0), (2, 0) and (4, 0). The graph looks like:



Example:

Write the equation of the graph shown.



Solution:

The x-intercepts are (-3, 0), (1, 0) and (2, 0). So the factors of the equation are x + 3, x – 1, and x – 2. Because the end behavior is left up and right down, the equation must have a negative leading coefficient and an odd degree. Therefore, the equation is P(x) = -(x + 3)(x – 1)(x – 2).

Real Zeros

If P(x) is a polynomial function in one variable, then the following statements are equivalent

- x = a is a *zero* or *root* of the function P(x).
- x = a is a *solution* of the equation P(x)=0.
- (x – a) is a *factor* of the function P(x).
- (a, 0) is an *x-intercept* of the graph of P(x).

It was stated earlier that if the polynomial has degree n then there are at most n real zeros. There is no claim made that they are all unique (different). Some of the roots or zeros may be repeated.

These are called **repeated roots**. Repeated roots are tied to a concept called multiplicity. The **multiplicity** of a root is the number of times a root is an answer. The easiest way to determine the multiplicity of a root is to look at the exponent on the corresponding factor.

For example, consider the function $P(x) = (x-3)^2 (x+5) (x+2)^3$, the roots to the function will be x=3 with multiplicity 2, x=-5 with multiplicity of 1, and x=-2 with multiplicity 3. If a zero has a multiplicity of 1, then it is unnecessary to write a multiplicity of 1 as it is assumed.

The multiplicity of a root, and likewise the exponent on the factor, can be used to determine the behavior of the graph at that zero.

- If the multiplicity is **odd**, the graph will **cross** the x-axis at that zero. That is, it will change sides, or be on opposite sides of the x-axis. If the multiplicity is one, then the graph will go straight through the zero. If the multiplicity is 3, the graph will wiggle (similar to x^3) as it goes through the zero.
- If the multiplicity is **even**, the graph will **touch** the x-axis at that zero. That is, it will stay on the same side of the axis. That is, for a multiplicity of 2, the graph will bounce off the axis and look parabolic around that zero.

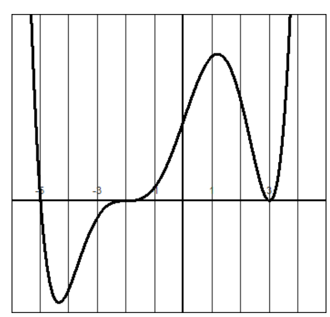
Examples:

Sketch the graphs of the following functions.

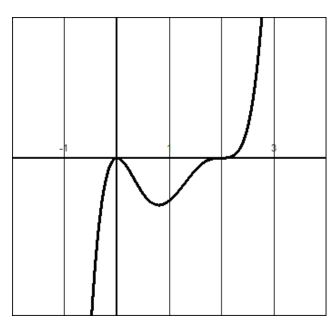
1. $P(x) = (x-3)^2 (x+5) (x+2)^3$
2. $P(x) = x^2 (x – 2)^3$
3. $P(x) = -x(x + 2)(x – 2)$

Solutions:

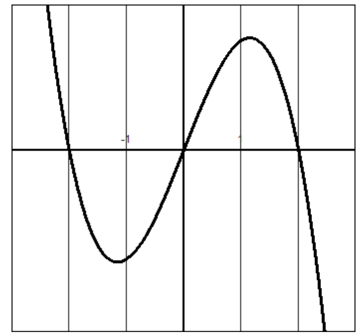
1. This graph has a positive leading coefficient and an even degree so the end behavior is both ends up. The x-intercepts are (3, 0) with multiplicity of 2 so the graph will bounce, (-5, 0) with multiplicity 1 so the graph will go through, and (-2, 0) with multiplicity of 3 so the graph will wiggle and go through.



2. This graph has a positive leading coefficient and an odd degree so the end behavior is left down and right up. The x-intercepts are (0, 0) with multiplicity of 2 so the graph will bounce, and (2, 0) with multiplicity of 3 so the graph will wiggle and go through.



3. This graph has a negative leading coefficient and an odd degree so the end behavior is left up and right down. The x-intercepts are (0, 0), (-2, 0), and (2, 0) all with multiplicity of 1 so the graph will go through all of them.



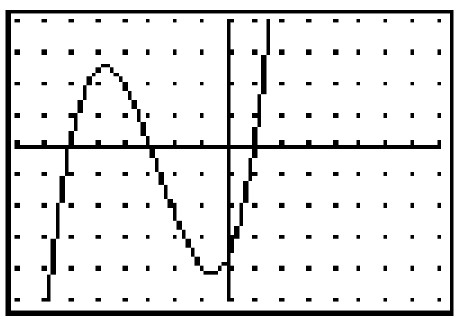
The calculator can be used to help factor higher order polynomials. We can use the ZERO key to find the zeroes of the function and then use those zeroes to write the polynomial in factored form. This works as long as all the zeroes are rational numbers or if there is only one pair of complex zeroes.

Examples:

1. Use the graphing calculator to write the polynomial $x^3 + 8x^2 + 9x - 18$ in factored form.

Solution:

Using the calculator to graph $Y_1 = x^3 + 8x^2 + 9x - 18$ in the window [-8, 8 ,1] by [-25, 20,5] gives

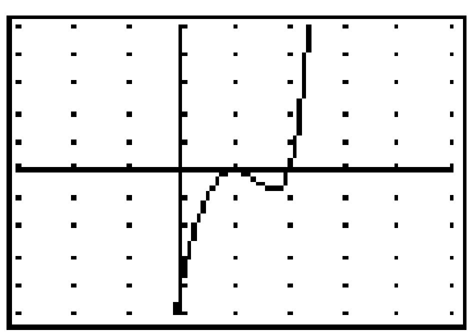


We can see by the graph that the zeroes are at (-3, 0), (-6 0) and (1, 0). Since it is a third degree polynomial and we have found 3 zeroes, we have found all the zeroes. Therefore, the polynomial in factored form is (x + 3)(x + 6)(x -1).

2. Use the graphing calculator to write the polynomial $x^5 - 4x^4 + 7x^3 - 10x^2 + 10x - 4$ in factored form.

Solution:

Using the calculator to graph $x^5 - 4x^4 + 7x^3 - 10x^2 + 10x - 4$ in the window [-3, 5, 1] by [-5, 5, 1]



We can only see two zeroes on this graph at (1, 0) and (2, 0). But the graph has a bounce at (1, 0) so that means that the zero has a multiplicity of 2. To find the other zeroes, we can use synthetic division. We divide by each of the three zeroes in succession. We should get a remainder of zero each time.

1⌋	1	-4	7	-10	10	-4
		1	-3	4	-6	4
1⌋	1	-3	4	-6	4	0
		1	-2	2	-4	
2⌋	1	-2	2	-4	0	
		2	0	4		
	1	0	2	0		

We can see by the synthetic division that the last factor is $x^2 + 2$. Therefore, the polynomial in factored form is $(x – 1)^2 (x – 2) (x^2 + 2)$.

This method works well for polynomials with rational zeroes but does not work well for irrational zeroes or polynomials with more than one set of complex zeroes. You could use the ZERO key to approximate the zeroes of the function to estimate the factors.

Homework:

List the degree and the leading coefficient of each of the following equations.

1. $P(x) = -11x^5 – 7x^3 + 4x^2 + 5x – 1$
2. $P(x) = 9x^6 – 7x^4 + 4x^3 + 2x^2 – 10x$
3. $P(x) = -8x^4 + 5x^3 + x^2 + 9x – 1$
4. $P(x) = -11x^6 – 7x^3 + 4x^2 + 5x – 1$
5. $P(x) = -2(x + 4)(x – 7)(x – 10)^2$
6. $P(x) = x^4(x + 3)(x – 5)(x – 6)$

Give the end behavior of the following equations.

7. $P(x) = -11x^5 – 7x^3 + 4x^2 + 5x – 1$
8. $P(x) = 9x^6 – 7x^4 + 4x^3 + 2x^2 – 10x$
9. $P(x) = -8x^4 + 5x^3 + x^2 + 9x – 1$
10. $P(x) = -11x^6 – 7x^3 + 4x^2 + 5x – 1$

11. $P(x) = -2(x + 4)(x - 7)(x - 10)^2$

12. $P(x) = x^4(x + 3)(x - 5)(x - 6)$

Find the x-intercepts of the following equations.

13. $P(x) = 2(x + 5)^2(x - 4)(x - 8)^2$

14. $P(x) = x^2(x - 3)(x - 5)$

15. $P(x) = -(x + 1)(x - 2)(x - 5)^3$

Sketch the following graphs.

16. $P(x) = -(x + 4)(x - 2)(x - 5)$

17. $P(x) = x^2(x - 4)^2$

18. $P(x) = 2(x + 5)^2(x - 4)(x - 8)^2$

19. $P(x) = x^2(x - 3)(x - 5)$

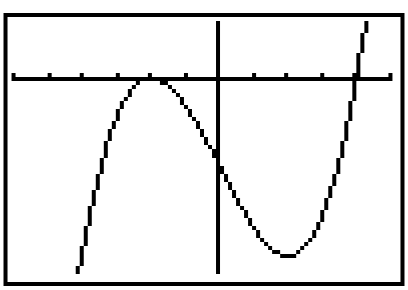
20. $P(x) = -(x + 1)(x - 2)(x - 5)^3$

21. $P(x) = -2(x + 4)(x - 3)(x - 7)^2$

22. $P(x) = x^3(x + 3)(x - 5)(x - 6)$

Give the factored form of the equation for the following graphs.

23.



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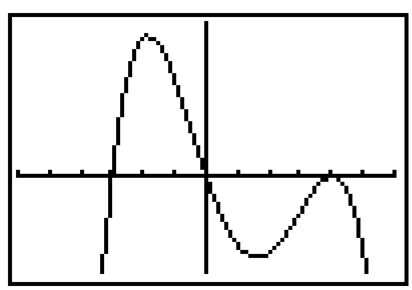

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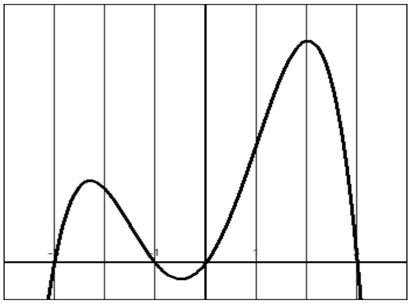

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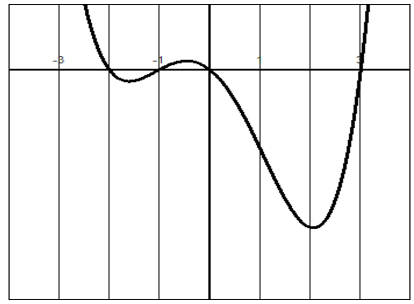

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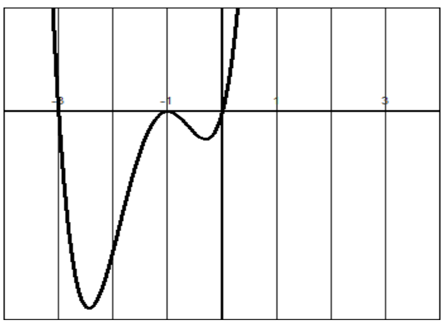

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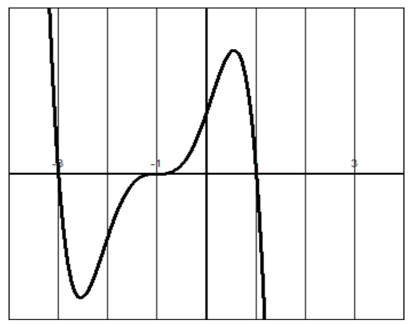


Use your graphing calculator to write the following polynomials in factored form.

29. $x^5 - 4x^4 - 4x^3 + 34x^2 - 45x + 18$

30. $x^4 - 7x^3 + x^2 + 63x - 90$

31. $x^3 - 3x^2 - 18x + 40$

32. $x^4 - 3x^3 - 14x^2 + 12x + 40$

33. $6x^3 + 17x^2 + x - 10$

7.5 Polynomial, Rational, and Absolute Value Inequalities

Previously, we solved quadratic inequalities. In this section, we will use similar techniques to solve polynomial and rational inequalities. To solve inequalities, we first need to determine boundary values and then we can use either test points or a graph to determine the intervals where the inequality is true. The solution set can be written in inequality notation or, more commonly, in interval notation.

Steps to solve an inequality:

1. Find the values which would solve the equation. These are the boundary values.
2. Using the graph, look in each interval between the boundary values to determine which intervals solve the inequality.
3. To solve without using a graph, choose a value within each interval between the boundary values to determine whether the interval solves the inequality. These results can be presented in a sign chart.
4. Write the solution as a union of all intervals which solve the inequality in interval notation.

A sign chart is a number line on which the boundary values are marked. A value in each interval is then tested to determine whether or not the interval solves the inequality and this is notated above the interval.

Examples:

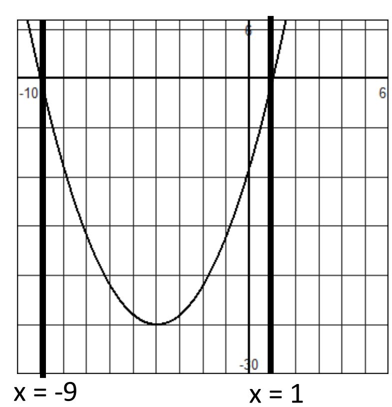
1. Solve $x^2 + 8x - 9 > 0$.

First solve $x^2 + 8x - 9 = 0$.

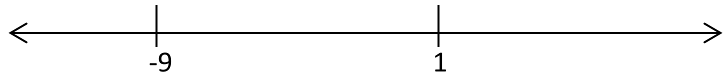
Factoring: $(x + 9)(x - 1) = 0$

The solutions are $x = -9$ and $x = 1$. These are the boundary values. Now, we can use a graph or a sign chart to solve the inequality.

Shown on the right is the graph of $f(x) = x^2 + 8x - 9$ with the boundary values marked on the graph. Looking to the left of $x = -9$, we can see that the graph is above the x-axis, so the interval values are greater than zero. Between the values $x = -9$ and $x = 1$, the graph is below the axis, so the interval is less than zero. In the interval to the right of $x = 1$, the graph is above the axis, so the interval is greater than zero. We were looking for where $x^2 + 8x - 9 > 0$. This occurs in $(-\infty, -9) \cup (1, \infty)$.

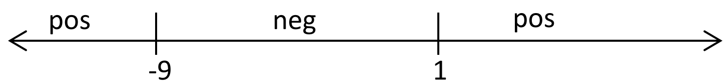


This problem could also be solved using a sign chart. The first step would still be to find the boundary values of x = -9 and x = 1. Then, we draw a number line and mark the boundary values.



Choose a number within each interval on the number line (test value) and plug it into the equation. Note whether the result is positive or negative.

Interval	Test value	$x^2 + 8x - 9$	Positive or negative?
(-∞, -9)	-10	11	Positive
(-9, 1)	0	-9	Negative
(1, ∞)	3	24	Positive



Recall that we are solving $x^2 + 8x - 9 > 0$, so the intervals where the result s positive form the solution. The solution is (-∞, -9) U (1, ∞).

2. Solve $x^3 - 2x^2 - 3x < 0$ symbolically.

First, we must solve $x^3 - 2x^2 - 3x = 0$. Factoring, we get:

$x(x^2 - 2x - 3) = 0$

$x(x - 3)(x + 1) = 0$

x = 0, x = 3, x = -1

The boundary numbers are x = -1, x = 0, and x =3. Now, let's create a sign chart.

Interval	Test value	$x^3 - 2x^2 - 3x$	Positive or negative?
(-∞, -1)	-2	-10	Negative
(-1, 0)	-1/2	0.875	Positive
(0, 3)	1	-4	Negative
(3, ∞)	4	20	Positive

To solve $x^3 - 2x^2 - 3x < 0$, we need the intervals which are negative. Thus, the solution is

(-∞, -1) U (0, 3).

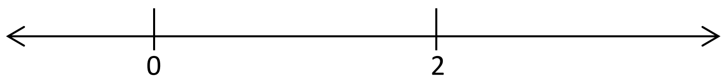
Rational inequalities are solved in the same way except that we must include all vertical asymptotes and holes as boundary values as well as values that make the equation equal to zero. Recall that vertical asymptotes and holes occur where the denominator is equal to zero.

Example:

Solve $\frac{2-x}{3x} > 0$.

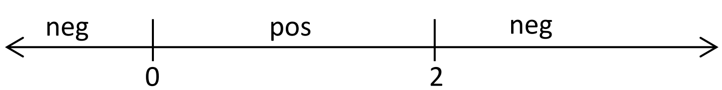
First we must find the boundary values. These occur where the numerator or the denominator equals zero.

$2 - x = 0$ gives $x = 2$ and $3x = 0$ gives $x = 0$. So the boundary values are $x = 2$ and $x = 0$. Make a sign chart.



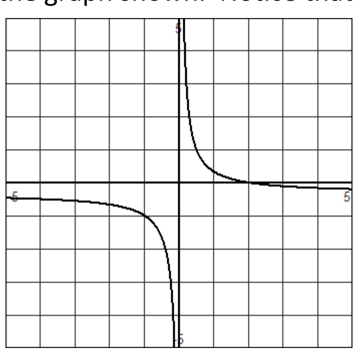
Choose a number within each interval on the number line (test value) and plug it into the equation. Note whether the result is positive or negative.

Interval	Test value	$\frac{2-x}{3x}$	Positive or negative?
$(-\infty, 0)$	-1	-1	Negative
(0, 2)	1	1/3	Positive
$(2, \infty)$	3	-1/9	Negative



To solve $\frac{2-x}{3x} > 0$, we are looking for intervals that are positive. Therefore, the solution is (0, 2).

This result can be verified by looking on the graph for intervals where the graph is above the x-axis. See the graph shown. Notice that the graph is above the x-axis on the interval (0, 2).



Solving absolute value inequalities

Recall that the definition of absolute value is $|x|$ is the distance of x from zero on a number line. So, $|x| = 3$ are numbers which are 3 units from zero which are 3 and -3. Now, let's look at $|x| < 3$. This inequality includes all numbers which are a distance less than 3 from zero which is the interval (-3, 3) or all numbers between -3 and 3 on the number line. Given the inequality $|x| < a$, the solution is always of the form $-a < x < a$. This holds true for any inequality in this form. Now, let's look at $|x| > 3$. This inequality includes all numbers which are a distance more than 3 from zero which are the intervals (-∞, -3) or (3, ∞) or all numbers further than -3 or all numbers further than 3 on the number line. Given the inequality $|x| > a$, the solution is always of the form $x < -a$ or $x > a$.

Steps to solve an absolute value inequality

1. Isolate the absolute value on one side of the inequality symbol.

2. If the symbol is < or ≤, then the solutions are of the form $-a < x < a$. This is an "and" statement.

   Therefore, write the problem without the absolute value signs between the negative and positive values of a. Solve the compound inequality.

3. If the symbol is > or ≥, then the solutions are of the form $x < -a$ or $x > a$. This is an "or" statement.

   Therefore, write the problem without the absolute value signs and solve the inequality. Also, write the problem without the absolute value signs, reverse the inequality sign and negate the value of a. Then solve that inequality. The solution includes both intervals.

Examples

1. Solve $|2x + 3| < 9$ algebraically. Verify graphically.

   Solution:

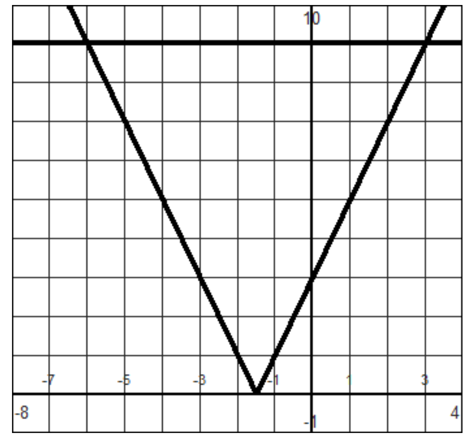
   The absolute value is isolated on one side so we can rewrite the absolute value inequality as a compound inequality since the symbol is <.

   $-9 < 2x + 3 < 9$

-12 < 2x < 6	Subtract 3 from each part of the inequality.
-6 < x < 3	Divide each part of the inequality by 2 to solve for x.
(-6, 3)	Write the solution in interval notation.

   To verify this solution graphically, we need to sketch the graph of y = $|2x + 3|$ and y = 9 on the same grid.

We are looking for where the absolute value graph is less than 9 or where the graph is below the line at y = 9. We can see the points of intersection are (-6, 9) and (3, 9). The absolute value graph is below the line y = 9 between x = -6 and x = 3 or on the interval (-6, 3).



2. Solve $|-4x+3| \geq 11$algebraically.

Solution:

The absolute value is isolated on one side of the inequality. Since the symbol is ≥, we rewrite the absolute value inequality as two inequalities and solve each.

-4x + 3 ≥ 11	or	-4x + 3 ≤ -11
-4x ≥ 8	or	-4x ≤ -14
x ≤ -2	or	x ≥ $\frac{14}{4} = \frac{7}{2}$
(-∞, -2]	or	$\left[\frac{7}{2}, \infty\right)$

3. Solve $2|3x-4|+6<4$ algebraically.

Solution:

We must first isolate the absolute value.

$2|3x-4|+6<4$

$2|3x-4|<-2$ Subtract 6 from both sides of the inequality

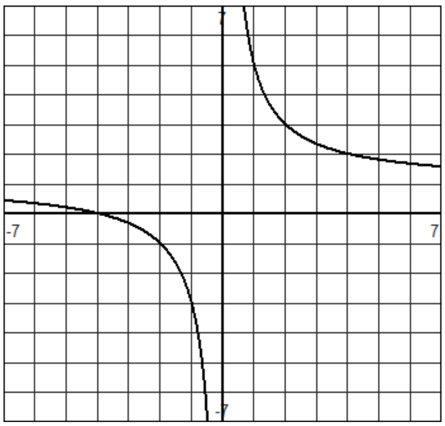
$|3x-4|<-1$ Divide both sides by 2

We can stop here because the absolute value will always be at least zero and so there are no possible values of x which can make the absolute value less than -1. Therefore, this inequality has no real solutions.

Homework:

Use the graphs shown to solve the equation and inequalities.

A. f(x) = 0
B. f(x) < 0
C. f(x) > 0

1.


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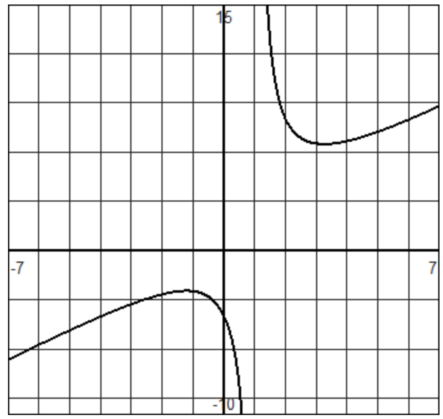

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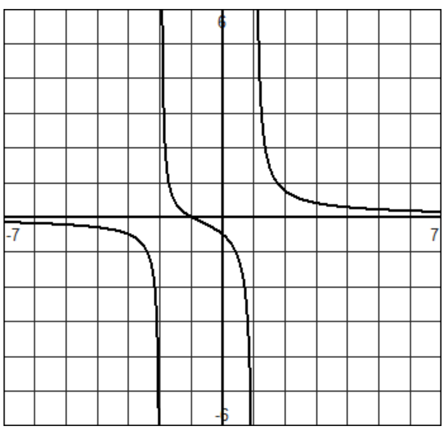

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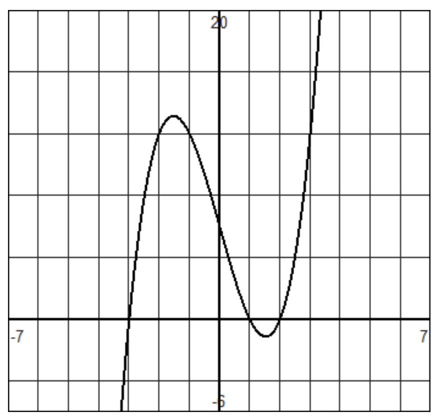

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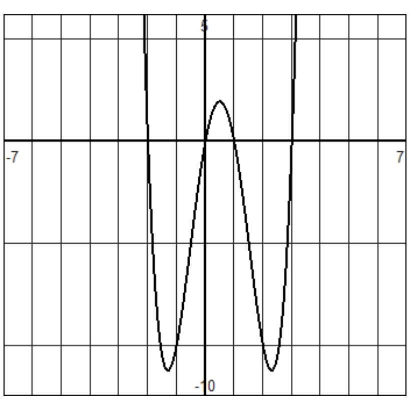

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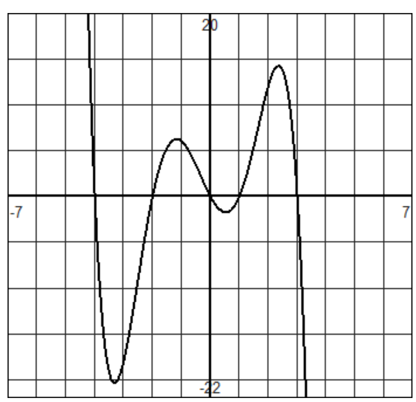
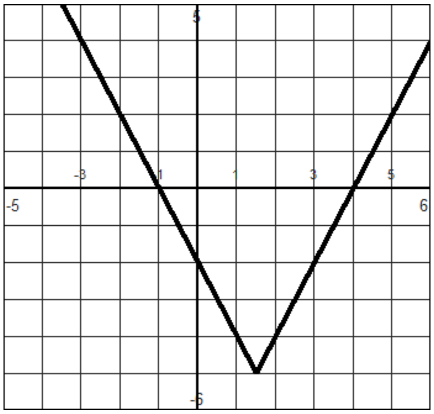
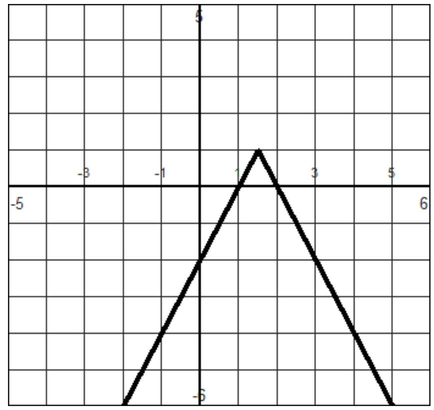
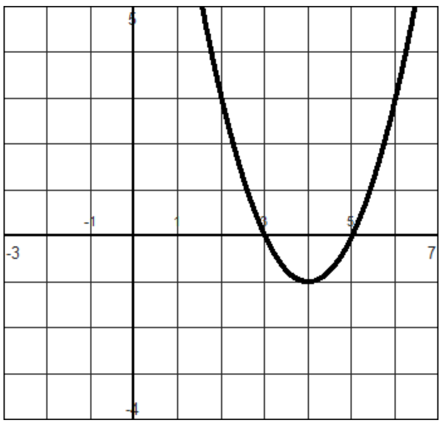
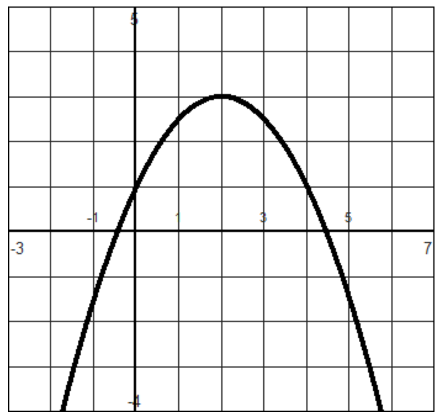


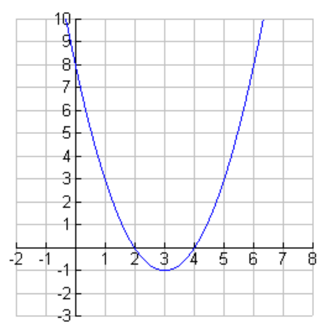
7. 

8. 

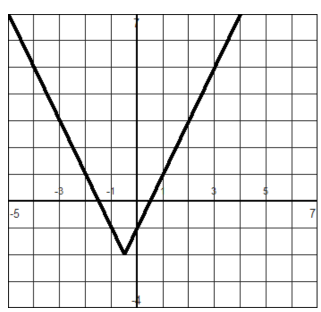
9. 

10. 

11. Use the graph to solve $x^2 - 6x + 8 < 3$.



12. Use the graph to solve $|2x + 1| - 2 > 5$



Solve the inequalities graphically.

13. $2x^2 - 5x + 2 \leq 0$

14. $-3x^2 < 2x + 7$

15. $x^3 - x < 0$

16. $x^3 + x^2 - 2x > 0$

17. $\frac{x+2}{x-1} \leq 0$

18. $\frac{x-1}{x^2-9} > 0$

19. $|3x + 4| < 2$

20. $|2x - 5| \geq 10$

21. $0.5\,|-x + 4| - 6 < 2$

Solve the inequalities symbolically using a sign chart.

22. $2x^3 - 3x^2 - 5x < 0$

23. $7x^4 - 28x^2 > 0$

24. $\frac{(x+2)^2}{x-5} \leq 0$

25. $\frac{x+3}{x^2-x} > 0$

26. $x^2 + 9x - 10 \geq 0$

27. $6x^2 + 11x - 10 < 0$

28. $(x + 1)(x - 5)(2x + 3) \leq 0$

Solve the inequalities algebraically.

29. $|2x + 15| \geq 10$

30. $3|2x - 5| + 1 > 10$

31. $3|x + 7| + 1 < 4$

32. $|6x - 5| + 4 \leq 2$

33. A ball is thrown across a field. The height of the ball at time t, in seconds, is given by $h(t) = -4.9t^2 + 15t + 2$ where h is in meters. At what times was the ball above 6 meters in the air?

34. The number of bacteria in a refrigerated food is given by $N(t) = 20T^2 – 20T + 120$, for $-2 \leq T \leq 14$ and where *T* is the temperature of the food in Celsius. At what temperature will the number of bacteria be less than 90?

35. The path of a high diver is given by $y = -\frac{4}{9}x^2 + \frac{24}{9}x + 10$ where y is the height in feet above the water and x is the horizontal distance from the end of the diving board in feet. How far from the board (horizontally) is the diver when he is less than 3 feet above the water?

36. At a certain company the average salary for a new graphic designer is $37, 500, but the actual salary could differ from the average by as much as $2570.
    A. Write an absolute value inequality to describe the situation.
    B. Solve the inequality to find the range of starting salaries.